How Adults Understand and Reason about Fractions

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Abstract

We investigated what strategies adults use to compare magnitudes of fractions, and how the strategies vary with overall mathematical knowledge. Because little is known about how people think about fractions, prior research has attempted to assess the extent to which people think of fractions compared to how they think about whole numbers (Bonato, Fabbri, Umilta & Zorzi, 2007; Schneider & Siegler, 2010; Meert, Gregoire, Noel, 2009). In particular, Bonato et al. (2007) has examined the degree to which fractional magnitude comparisons can yield an understanding of the mental representation of fractions. Schneider & Siegler (2010) argue that these assessments cannot be made independently from understanding the strategies for comparison that a particular pair of fractions elicits. The current study extends prior research by comparing fraction magnitude comparison strategies of students at a selective university, with high math proficiency, with those of students at a community college, with lower math proficiency. The goals of this thesis were to identify, explain and define the strategies that are used in fraction magnitude comparisons by adults, investigate how these strategies vary with the math proficiency of the adults, and evaluate whether adults who do not consistently use desirable strategies recognize desirability of good alternative strategies. Our findings indicate that strategy use and consistency of using good and poor strategies vary with overall math knowledge and performance on our magnitude comparison task; that lower performing participants more frequently used strategies that would yield incorrect results and less often recognize when to switch to good alternative strategies compared to the high performing participants.

Introduction

The study of how people process and mentally represent whole numbers has generated a plethora of theories and information for several decades. Recently, this research has begun to include questions about how information generated from whole number research applies to fractions. One particularly prevalent phenomenon found with whole numbers is the distance effect. As first explained by Moyer and Landauer (1967), when asked to compare the relative magnitudes of two whole numbers, people determine which number is larger faster and more accurately when the numbers are very far apart compared to when the numbers are closer together. For example, when asked to solve a comparison between 10 and 50, people will be faster and more accurate to say that 50 is larger compared to solving a comparison with closer numbers like 45 and 50. Speed to solve such problems has been found to follow a logarithmic function, such that as distance between magnitudes increase, solution times decrease, a phenomenon that has been attributed to the presence of a mental number line (Dehaene, Dupoux, & Mehler, 1990).

Numerical cognition researchers have extended the investigation of the distance effect to fractions in an effort to gather more information about the nature of the mental representation of fraction magnitudes. Recent research has provided evidence for and against the presence of the distance effect in fraction magnitude comparisons. Bonato, Fabbri, Umilta and Zorzi (2007) asked adults to compare fractions and say which was larger. Bonato et al. found that there was no evidence to suggest that there was a distance effect when comparing fraction magnitudes. The solution time to solve the comparisons did not follow a logarithmic function, such that solution times decrease as the distance between the magnitudes increases. Rather, they found that solution times were related to the distances of the whole number parts of the fractions (ie., distance between the denominators). Bonato et al. posit that these findings suggest that adults were using componential strategies to compute the magnitude comparisons.

However, further investigation of the stimuli used in this study yields questions about this interpretation of the Bonato et al. (2007) findings. While there was no logarithmic pattern in the solution times in relation to the distance between the magnitudes, the stimuli were not set up in such a way that would elicit this pattern. In all but one experiment, the fraction comparisons consisted entirely of fractions where the numerators were equal (ie. 1/8 vs. 1/5) or where the denominators were equal (3/4 vs. 1/4). The remaining experiment included only problems where either the numerator or the denominator was equal. This comparison does not require the participant to represent the magnitude of the fraction. Rather, the participant need only consider the magnitudes of the denominator, the magnitude of the numerator, or first one and then the other. On 1/8 vs. 1/5, for example, 1/8 has a larger denominator and thus the overall fraction's magnitude is smaller than 1/5.

A more compelling case testing the presence of the distance effect in fraction magnitude comparisons would involve fractions that require the participants to process and compare the integrated magnitudes of the fractions. If participants do not indeed process and represent fraction magnitudes, the distance effect would not be present. However, this cannot be concluded from stimuli that only require partial comparisons to generate correct responses.

Other studies have been conducted that show support for the distance effect in fraction magnitude comparisons. Schneider and Siegler (2010) found that among students at an elite university and at a community college, the distance effect is present in fraction magnitude comparisons that require processing of the fraction's integrated magnitude rather than the individual magnitudes of the fraction's component parts. While the accuracy is lower and solution times slower at the community college in comparison to the elite university, the results were still consistent with the presence of a distance effect at both schools.

There is also other evidence that results of a fraction magnitude comparison may yield differing findings about internal mental representations of fraction magnitude depending on the context of the question. Meert, Gregoire, and Noel (2009) conducted a study that involved a fraction magnitude comparison but with a slightly different paradigm. In this study, Meert and colleagues mixed fraction comparisons that only involved equal numerators or equal denominators with whole number comparisons. The fraction magnitudes were supposed to serve as a prime for the whole number comparisons. For example, the participants saw an equal numerator comparison x/a vs. x/b followed by the comparison of a vs. b or they saw an equal denominator comparison a/x vs. b/x followed by a vs. b. Equal numerator comparisons could be solved componentially by selecting the smaller denominator, which is associated with the larger fraction when numerators are equal. Meert et al. argued that if the participants were using a componential strategy, they would then be slower to solve a whole number comparison with the denominator values from the fraction comparison. Conversely, on the equal denominator comparisons, participants need to locate the larger numerator. On the subsequent whole number comparison, participants should be faster to solve the comparison because the numbers were primed on the previous fraction comparison. They found that the solution times reflected these patterns. However, as in the Bonato et al. (2007) findings, these comparisons might be easier to solve using these componential strategies. This does not suggest that on other types of problems a holistic strategy would not be used. Overall, Meert et al. (2009) suggest that fraction magnitude representation may be a hybrid of componential and holistic, depending on the context.

Other research provides support for the idea that processing of fractions is context dependent. Measures of automatic processing of number magnitudes such as the size congruity effect and the semantic congruity effect suggest that the processing of fractions as an integrated unit is not automatized (Kallai & Tzelgov, 2009; Ganor-Stern, Karasik-Rivkin & Tzelgov, 2011). The size congruity effect refers to the finding that people are faster to say that a number is larger if its numerical value and physical size, relative to the other number, are both larger than if the numerical value and physical size are smaller than the other number. Numerical value is defined as the size of the magnitude of the fraction. Physical size is defined as the size of the numbers of the computer screen or piece of paper (ie. font size).

Kallai & Tzelgov (2009) found that when the comparisons involved unit fractions (1/N), an inverse size congruity effect was found. Participants were faster in responding when the fractions with smaller denominators and therefore larger numerical size were physically smaller than the opposing, numerically smaller but physically larger fraction. However, when the comparisons included improper fractions and non-unit fractions, no size congruity effect was found; solution times were not faster depending on the match of numerical size and physical size of the fraction.

Further emphasizing the importance of context on comparison strategies, Ganor-Stern, Karasik-Rivkin & Tzelgov (2011) found that when you include 0 and 1 in some of the fraction magnitude comparisons, this yields a holistic representation of magnitude comparisons with unit fractions. When the comparison included 0 or 1, there was a distance effect for overall fraction magnitude in solution times, and when the comparison only involved unit fractions, there was evidence of the semantic congruity effect. The semantic-congruity effect refers to the finding that comparisons between numbers that are large in the experimental context are faster when the task instructions require the participant to select the larger number, and comparisons between numbers that are small in the experimental context are faster when the instructions requiring the selection of the smaller number. These findings suggest that contextual aspects like size of the fraction or semantics of the task instructions may change how people are processing and solving the fraction magnitude comparisons because of the differential solution time results.

Research in the domain of neuroscience and neuroimaging has provided evidence that fractions may be processed holistically as an integrated magnitude. Single cell recording techniques have demonstrated that rhesus macaques have specific sets of neurons that are responsible for coding a specific absolute number magnitude (Nieder, Freedman, & Miller, 2002; Nieder and Miller, 2003). In addition, the intraparietal sulcus has been shown to be associated with an analog representation of whole number magnitude in the human brain (Nieder 2005).

Jacob and Nieder (2009) extended this research to fractions using a functional MRI adaptation experiment. The authors found that there are regions of neurons in the human fronto-parietal cortex that code for integrated magnitudes of fractions across formats of presentation. For example, the region responds to ½ and "half" in the same way. That is, we have the same pattern of activation when we see the word for a fraction value or see the symbolic value written out. Additionally, in a magnitude comparison task, they found evidence for the distance effect for number and word fractions. These results lend support to the idea that fractions are not necessarily processed as separate numerator and denominator but rather an integrated magnitude.

Ischebeck, Schocke, and Delazer (2009) also conducted an fMRI study that involved a fraction magnitude comparison. Ischebeck et al. (2009) found behavioral evidence for the distance effect in the difference in the overall fraction magnitude and the differences between numerators of the fractions and the denominator of the fractions. However, the activation in the intraparietal sulcus as measured by the fMRI data was most strongly correlated to the distance between overall fraction magnitude. This again provides evidence for an integrated magnitude representation of fraction magnitude.

Another area of research that provides information about how children and later adults process and understand numerical magnitudes other than whole numbers is the study of understanding of decimal fractions. Elementary school children and even some adults have been shown to have several misconceptions about decimal fractions (Kouba, Carpenter, & Swafford, 1989; Putt, 1995). These misconceptions are generally the result of children overgeneralizing certain characteristics of whole numbers to decimal fractions, such as the idea that longer fractions are larger (.607 is larger than .9 because 607 has 3 digits and 9 has only 1) (Stacey & Steinle, 1999; Resnick et al., 1989). Rittle-Johnson, Siegler, and Alibali (2001) conducted an intervention with children in order to improve their conceptual and procedural understanding of decimal fractions. They found that when given an intervention that correlates the decimal fraction notation with magnitude of the number on a number line, both conceptual and procedural understanding of decimal fractions greatly increased.

This suggests that children benefit greatly from the reinforcement of magnitude of the decimal fractions. It also suggests that children have the ability to represent non whole number units on a mental number line. Fractions are essentially the same thing as decimals, with the exception that the notation is different. Fractions involve two sets of whole numbers that need to be integrated; decimals indicate the result of dividing the numerator by the denominator. Fractions and decimals represent two different ways of representing rational numbers. Thus decimals are essentially the integrated magnitude of a fraction. It follows that adults would be able to represent integrated magnitudes of fractions, as there is evidence that children are able to represent these decimal fraction magnitudes.

The results of these various studies suggest that adults may employ whichever type of representation is most advantageous in a given situation. They indicate that the presence of the distance effect depends on whether the stimuli encourage or require an integrated, analog magnitude representation of the fraction magnitude. Overall, there is reason to believe that while adults are capable of representing fraction magnitudes in this integrated manner, they do not always do this if they do not have to.

Adults may engage in a componential strategy, in which only component parts of the fractions are compared, in cases where this strategy would yield the lowest cognitive cost (Meert, Gregoire, & Noel, 2010). In other cases where this strategy may not yield the correct result, adults may make use of other strategies, including an integrated magnitude representation of the fraction (Schneider & Siegler, 2010). Thus, an investigation into what types of strategies adults use in fraction magnitude comparisons can help to elucidate how these fractions are being processed and represented.

Strategy Use in Math Problem Solving

A widely accepted notion in numerical cognition research is that both adults and children use discrete strategies to solve mathematical problems especially arithmetic problems (Shrager & Siegler, 1998, Lemaire & Reder, 1999; Schunn & Reder, 2001; Crowley, Shrager & Siegler, 1997). These discrete strategies can give insight into solution time and accuracy results for mathematical problem solving. Specifically, if solution time and accuracy are averaged over trials where different strategies were used, this can lead to a misinterpretation of the data

(Siegler, 1987). Thus, in the case of fraction magnitude comparisons, where there is evidence to suggest that different strategies are used, the accuracy and solution time results should be considered separately based on the discrete strategies used on the particular trial or comparison.

Additionally, there is evidence that suggests that strategies are selected adaptively for the particular problem being considered. In arithmetic, Siegler and Lemaire (1997) found that when given the option to solve multi-digit multiplication problems using mental calculation, using a calculator, or using pencil and paper, the strategy chosen was dependent on the specific features of the problem. In particular, the relative speeds and accuracies of the strategies were closely related to how often each strategy was used on a particular problem.

While children and adults actively select strategies that are beneficial to solving a particular problem, there are many individual differences in how strategies are chosen and what strategies are chosen. Siegler (1988) found that first graders engaged in addition, subtraction, and reading tasks used different strategies that varied based on the child's individual knowledge of the problem and the child's threshold for stating answers. Further, there were three types of strategy users: good students (good knowledge, low threshold), not-so-good students (less good knowledge, low threshold) and perfectionists (good knowledge, high threshold). This suggests that strategy use varies with both background knowledge and individual answer threshold. Additionally, the feedback of the environment may play a role in the individual differences in strategy use. Schunn and Reder (2001) found that while people may have the same ability to adaptively select among strategies for a particular problem, this ability may vary with the particular feedback of the environment.

While strategy use has been widely studied in whole number arithmetic, there has not been much research in terms of strategies used in problem solving with fractions. Stafylidou and

Vosniadou (2004) found that children use strategies to solve fraction problems that relate three basic understandings of fractions: 1) fractions as two independent numbers, 2) fractions as parts of a whole, 3) a ratio relationship between numerator and denominator that can be bigger, smaller, or equal to a whole. Children usually exhibit performance that suggests that they have one of these basic conceptual understandings. However, they might also use strategies that suggest that they have a combination of these conceptual frameworks. There have not been any studies that have explicitly examined strategy use in fraction magnitude comparisons by having participants explicitly explain their strategy use.

The current research seeks to examine more closely the strategies used with fractions. Using self-report, college students explained strategies for particular fraction magnitude comparison problems. Self-report has been shown to be a valid measure of strategy use (Campbell & Alberts, 2009). Overall, this research seeks to gain a better understanding of how adults understand, process, and represent fractions. Additionally, it seeks to examine how the strategies of more and less mathematically proficient adults differ on fraction magnitude comparisons.

Goals of the Current Research:

The three experiments described in this paper have three central goals. The first is to enumerate, explain, and define the strategies that are used in fraction magnitude comparisons by adults. This goal is intended to produce a clearer picture of the basis of fraction magnitude representation, as well as how adults might use specific strategies that are adaptive to the problem type. The second goal is to investigate how these strategies vary with the overall math proficiency of the adults. The third goal is to evaluate whether adults who do adopt inferior strategies recognize the greater desirability of good alternative strategies.

Each of the three experiments addresses one of these goals. The first experiment asks undergraduates at a highly selective university to solve a variety of fraction magnitude comparison problems and then self report their strategy on each trial. The second experiment extended the first experiment to college students in order to evaluate the differences in performance and strategy use between students at an elite university and students at a community college. Based on some of the incorrect strategies that were used in the second study, the third study investigates whether students recognize they are using poor strategies and will switch to a better, correct alternative if one is provided. This study involves first asking community college students to complete the same fraction magnitude problems and report their strategies in the same way as in the first two studies, and then re-presenting the problems and asking whether the participant would use the same strategy as before or one of two other strategies (one that would yield the correct answer and one that would yield an incorrect answer).

Experiment 1

Methods

Participants

The participants were 19 undergraduate students (11 female) taking introductory psychology courses at a highly selective university. The median age was 20. The mean self-reported math SAT score was 720 (with a range of 620 to 800 out of a possible 800). This average score is consistent with the university's actual mean math score of 730 with a range of 620 to 800 (based on figures from the university's Admissions Department).

Design

The study used a within-subjects design. Participants were presented eight types of problems: equal denominator problems, equal numerator problems, larger numerator and smaller denominator problems, halves reference problems, multiply to produce common denominator problems, multiply to producer common numerator problems, magnitude estimation problems with distances greater than 33%, and magnitude estimation problems with distances less than 33%. For each type of problem, there were three items that had denominators between 2 and 9 inclusive and three items that had denominators between 11 and 19 inclusive. Thus, there were a total of 48 fraction magnitude comparisons (8 types of problems; 6 problems of each type). All of the fractions used in the study were between 0 and 1. See Table 1 for examples of stimuli from each problem type.

The stimuli were counterbalanced so that, excluding ties in the numerator or denominator, half of the larger fractions had numbers in the numerator and denominator that were smaller than the opposing fraction's numerator and denominator, and half of the larger fractions had numerators and denominators that were larger than the opposing fraction's numerators and denominators. Except for the halves referencing condition, both fractions in each comparison were on the same side of ½. The study was conducted using EPrime to measure accuracy and solution time for the responses. Additionally, Audacity was used to create audio recordings of the sessions in order to capture voiced descriptions of strategy use.

Procedure

Participants were instructed to estimate which of two fractions was larger. The two fractions appeared on the screen, one on the right side and one on the left side. Participants pressed a key on the same side of the keyboard, as the fraction they thought was larger to indicate which fraction was larger. The comparisons were counterbalanced so that half of the time the larger fraction appeared on the right side of the screen and half of the time the larger fraction appeared on the left side of the screen. Solution time and accuracy were collected after this response.

After the participant entered which fraction was larger, the program paused. The problem remained on the screen, and instructions appeared that told participants to explain the strategy that they used to estimate the larger fraction. Participants then explained aloud how they arrived at their answer. This information was coded later based on the audio recordings.

This two-step procedure was repeated for each of the 48 fraction comparisons. The stimuli were shown in random order such that a comparison from each of the eight problem types was shown before a second problem from each group were shown and so forth.

In addition to the fraction magnitude comparison task, participants were also asked to complete a number line estimation task. This task consisted of giving the participants a paper with a number line with 0 at the left end and 5 at the right end, and a fraction at the top of the page that ranged from 0 to 5. Participants then had to place a hatch mark on the line where the fraction would go. The fractions chosen and the procedure used were modeled after Siegler et al. (2011). Four problems were chosen from each fifth of the number line: 1/19, 3/13, 4/7, 8/11, 7/5, 13/9, 14/9, 12/7, 13/6, 19/8, 8/3, 11/4, 13/4, 10/3, 17/5, 7/2, 17/4, 13/3, 9/2, and 19/4.

The fraction magnitude comparison task and the number line estimation task were counterbalanced so that half of the participants completed the number line estimation task first and half of the participants completed the magnitude comparison task first.

Results

Number line estimation

Accuracy. Accuracy of number line estimation was indexed by percent absolute error (PAE), defined as: *PAE* = (|Participant's Answer – Correct Answer|)/ Numerical Range X 100. For example, if a participant was asked to locate 5/2 on a 0 – 5 number line, and marked the location corresponding to 3/2, the PAE would be 20% ((|1.5-2.5|)/5 X 100). Note that PAE varies inversely with accuracy such that the higher the PAE, the less accurate the estimate.

Mean PAE for the participants was 6% (SD = 2.3). The distribution of PAE by participant is shown in Figure 1. The mean PAE for each participant was correlated with the overall accuracy on the Magnitude Comparison task such that as PAE decreases, accuracy on the magnitude comparison task increases (r = -0.48; t= -2.24; p = -.04). However, the mean PAE does not correlate with the self-reported Math SAT scores (r = -0.04; t = -0.17; p = 0.86).

Linearity. The equation for the linear best-fit line for the group mean estimate versus the actual magnitude was virtually perfect: Y = 1.005X - 0.12 (R^2 = .99). For individual participants, the best fitting linear function accounted for 95% percent of the variance. The results of the number line estimation task indicate that the participants have knowledge about the magnitudes of fractions that is consistent with a linear internal magnitude representation.

<u>Magnitude Comparison</u>

Accuracy. The mean overall accuracy per participant on this task was 97% (SD = 4). Mean accuracies for each of the problem types can be found in Table 2. A one way ANOVA examining differences in mean accuracies by problem types gives an overall difference between problem type (F(7, 144) = 3.59; p = 0.001). The accuracy for the Multiply for Common Numerators problem type was lower than all of the problem types except for the Halves Referencing and the Estimate Magnitudes (less than 1/3 apart) problem types: (Multiply for Common Numerator vs. Equal Denominators: (89% vs. 100%: F(1, 38) = 3.85; p < 0.001), Multiply for Common Numerator vs. Equal Numerator: (89% vs. 98%: $F(1,38) = 3.20$; $p < 0.05$), Multiply for Common Numerator vs. Estimate Magnitudes (more than 1/3 apart): (89% vs. 98%: $F(1,38) = 3.20$; $p < 0.05$), Multiply for Common Numerator vs. Larger Numerator and Smaller Denominator (89% vs. 100%: $F(1, 38) = 3.85$; $p = 0.005$), and Multiply for Common Numerator vs. Multiply for Common Denominator: (89% vs. 97%: $F(1,36) = 4.51$; $p = 0.04$).

Solution times. A one way ANOVA examining differences in mean solution times by problem type gives an overall difference between problem type ($F(7, 144) = 7.37$, $p < 0.001$). The solution times on the Equal Denominators problem type were faster than the Estimate Magnitudes (less than 1/3 apart) problem type (2.93 vs. 6.99: $F(1, 36) = 4.49$; $p < 0.001$), the Larger Numerator and Smaller Denominator problem type (2.93 vs. 3.55: $F(1, 36) = 3.82$, $p = 0.006$), the Multiple for a Common Numerator problem type (2.93 vs. 8.41: $F(1, 36) = 5.45$, $p < 0.001$), and the Halves Referencing problem type (2.93 vs. 7.74: $F(1, 36) = 5.07$, $p < 0.001$). Additionally, solutions times on the Equal Numerators problem type were faster than the Multiply for a Common Numerator problem type (4.49 vs. 8.41: $F(1, 35) = 3.74$, $p = 0.007$) and the Halves Referencing problem type (4.49 vs. 7.74: $F(1, 37) = 3.36$, $p = 0.03$). Solution times on the Estimate Magnitudes (greater than 1/3 apart) problem type were faster than the Multiply for a Common Numerator problem type (4.19 vs. 8.41: $F(1, 36) = 3.90$, $p = 0.004$) and the Halves Referencing problem type (4.19 vs. 7.74; $F(1, 36) = 3.52$, $p = 0.016$).

Mean accuracy and mean solution time on these problem types has a near perfect negative correlation ($r = -0.96$; $t = -8.9$; $p < 0.001$). Responses for the Equal Denominators problem type had the fastest solution times and had the highest possible accuracy (100%). Responses for the Multiple for a Common Numerator problem type had the slowest solution times and were the least accurate.

Strategies. In order to analyze the types of strategies that the participants were using on the different comparisons, an experimenter listened to the audio recordings trial by trial and coded the strategies as one of the 7 expected strategies or as another strategy. A complete list of the types of strategies that participants in Experiment 1 used can be found in Table 3. Strategies were grouped together into four overall categories: 1) strategies that must be correct by logical necessity ("Logical Necessity Strategies") 2) strategies that must be correct when intermediate steps executed correctly ("Intermediate Steps Strategies"), 3) strategies that should yield better than chance results when steps are executed correctly, but that won't necessarily be correct ("Usually Correct Strategies"), 4) strategies not guaranteed to yield above chance performance ("Questionable Strategies").

A one way ANOVA examining percent use of each of the four types of strategies revealed an overall difference in percent use of the strategy types ($F(3, 72) = 38.67$; $p < 0.001$). Strategies of the first two types were used most often. Logical Necessity strategies were used on 25% of trials, Intermediate Steps strategies were used on 49%, Usually Correct strategies on 12%, and Questionable strategies on 13% (Figure 4). Comparisons revealed that the Intermediate Steps strategies were used more than each of the other strategy types: the Logical Necessity strategies (49% vs. 25%: ($F(1, 36) = 6.18$, $p < 0.001$), the Questionable strategies (49% vs. 13%: ($F(1, 36) = 9.27$; $p < 0.001$), and the Usually Correct strategies (49% vs. 12%: $F(1, 36) = 9.38$; $p < 0.001$). Additionally, the Logical Necessity strategies were used more than the Questionable strategies (25% vs. 13%: $F(1,36) = 3.09$; $p = 0.02$) and the Usually Correct strategies (25% vs. 12%: $F(1, 36) = 3.20$; $p = 0.01$).

A one-way ANOVA revealed differences among strategies in accuracy ($F(3, 70) = 12.54$; $p< 0.001$). Consistent with their definition, the Questionable strategies had lower accuracy (80%

SD = 16) compared to the other 3 categories: Logical Necessity vs. Questionable (100% vs. 80%: $F(1, 37) = 5.21$; $p < 0.001$), Intermediate Steps vs. Questionable (98% vs. 80%: $F(1, 36) = 4.91$; $p < 0.001$), and Usually Correct vs. Questionable (99% vs. 80%: $F(1, 36) = 4.80$; $p < 0.001$). The other three strategies were non different and all near 100% accuracy (Figure 5).

A one way ANOVA examining the differences in solution time by strategy type revealed an overall difference between solution time by strategy type ($F(3, 70) = 8.9$; $p < 0.001$). The Questionable strategies had slower solution times than the other 3 categories: Logical Necessity vs. Questionable (4.32 vs. 8.62: $F(1, 37) = 4.72$; $p < 0.001$), Intermediate Steps vs. Questionable (5.27 vs. 8.62: $F(1,35) = 3.60$; $p = 0.004$), and Usually Correct vs. Questionable (4.35 vs. 8.62: $F(1, 36) = 4.00$; $p < 0.001$). The other three categories had solution times that were around 5 seconds (Figure 6). This suggested that these mathematically sophisticated students might have tried other strategies first, and only resorted to these strategies when they could not execute other, more promising ones.

Discussion

This experiment demonstrates both the linear internal representation of fraction magnitude and the types of strategies that participants use on a fraction magnitude comparison task. The participants showed a linear magnitude representation on the number line estimation task. This, together with the low mean PAE, suggests that these university students had a very accurate internal representation of the magnitudes of fractions, at least those ranging from 0 to 5.

Additionally, participants had high overall accuracy on the magnitude comparison task, which suggests that they were very well equipped to solve the magnitude comparison problems. The strategies that they used were encompassed by our predicted list of strategies, although some anticipated strategies were not observed, in particular converting to decimals or percents, visualization, and choosing the fraction with the smaller difference between its numerator and denominator.

Participants in large part used strategies that would yield accurate results and were very accurate at the task. To understand the strategies that participants with a wider range of ability on this task use, we repeated the study at a local community college in Experiment 2.

Experiment 2

Methods

Participants

Participants were 19 community college students (15 female, median age 21) who were recruited from introductory psychology courses and introductory history courses. The mean self-reported SAT math score for the 11 participants who indicated their scores was 503, with a range of 420 to 720. Only one participant reported a score of over 540; thus, there was minimal overlap with the self-reported SAT scores of the students from the Highly Selective University.

Design and Procedure

The design and procedure were identical to those of Experiment 1.

Results

The distribution of accuracy on the magnitude comparison task was bimodal. There were 10 participants who had a mean percent accuracy of 93%; their accuracy ranged from 88% to 98% correct. The other 9 participants had a mean accuracy of 52% and ranged from 15% to 79%

correct (Figure 3). The results have been separated into two groups based on this distribution: High Performing and Low Performing participants.

Number line estimation

Accuracy. The mean PAE for all participants on this task was 23% (*SD* = 13). A one way ANOVA examining the difference in PAE between the participants from the Highly Selective University in Experiment 1 and those from the Community college in Experiment 2 indicates that the PAE for the Highly Selective University students were lower than the Community college students (6% vs. 23%: $F(1, 36) = 28.69$; $p < 0.001$).

Within the Community College sample, the High Performing participants had a mean PAE of 12% (*SD* = 4), and the Low Performing participants had a mean PAE of 35% (*SD* = 9). A one way ANOVA examining the difference in PAE between High Performing and Low Performing participants showed that the High Performing participants PAE was lower than the Low Performing participants (12 vs. 35: $F(1, 17) = 56.62$; $p < 0.001$). This indicates that the High Performing participants had more accurate estimates compared to the Low Performing participants.

Linearity. The best fitting linear equation mapping the fractions onto the High Performing participants' group mean estimates was $Y=0.89X - 0.02$ ($R^2 = .95$), Figure 2. Analyses of individual performance of these participants demonstrated that the best fitting linear function accounted for an average of 82% of the variance in PAE.

The linear best-fit of the Low Performing participants' group mean estimate versus actual magnitude was $Y = 0.34X + 1.5$ ($R^2 = 64\%$) (Figure 2.) Analyses of individual performance indicated that the best fitting linear function accounted for 28% of the variance for members of this group. This was lower than the variance accounted for by the best fitting linear function of the individual performance of the High Performing participants (28% vs. 82%: $F(1, 17) = 16.3$; $p < 0.001$).

Magnitude Comparison Task

Accuracy. In order to compare the performance of the participants from the Highly Selective University (Experiment 1) and the Community college (Experiment 2), a 2: (Highly Selective University vs. Community College) X 8: (Problem type) ANOVA was used to examine differences between accuracy. Students from the Highly Selective University were more accurate than Community College students (97% vs. 74%: $F(1, 36) = 78$; $p < 0.001$). However, there were no differences based on accuracy by problem type ($F(7, 288) = 1.32$; $P = 0.24$) and there was no interaction between students at the Highly Selective University and students at the Community College and between problem type ($F(7, 288) = 0.49$; $p = 0.84$).

Solution Time. A 2: (Highly Selective University vs. Community College) X 8: (Problem type) ANOVA was used to analyze the differences in solution time between the students at the Highly Selective University and the Community College and on problem types. Students at the Highly Selective University were faster than Community College students (5. 92 s vs. 10.12 s: $F(1, 36) = 58.60$; $p < 0.001$). Additionally, there were overall differences in solution times by problem type ($F(7, 288) = 6.44$; $p < 0.001$). There was no interaction between students at the Highly Selective University vs. Community College students and problem type ($F(7, 288) = 0.11$; $p = 1.00$).

Mean solution times to the Equal Denominators problem type were faster than the Estimate Magnitudes (less than 1/3 apart) problem type (5.15 s vs. 9.15 s: $F(1, 75) = 3.64$; $p = 0.009$), than the Larger Numerator and Smaller Denominator problem type (5.15 s vs. 8.72 s: $F(1, 75) = 3.26$; $p = 0.04$), than the Multiply for Common Numerator problem type (5.15 s vs.

10.52 s: $F(1, 75) = 4.89$; $p < 0.001$), and than the Halves Referencing problem type (5.15 s vs. 10.20 s: $F(1, 75) = 4.60$; $p < 0.001$). Responses to the Equal Numerator problem type were faster than the responses to the Multiply for Common Numerator problem type (6.19 s vs. 10.52 s: $F(1, 75) = 3.95$; $p = 0.003$) and than the responses to the Halves Referencing problem type (6. 19 s vs. 10.2 s: $F(1, 75) = 3.66$; $p = 0.008$). Additionally, responses to the Estimate Magnitudes (greater than 1/3 apart) problem type were faster than the responses to the Multiply for Common Numerator problem type (6.36 s vs. 10.52 s: $F(1, 75) = 3.79$; $p= 0.005$) and than the responses to the Halves Referencing problem type (6.36 s vs. 10.20 s: $F(1, 75) = 3.50$; $p = 0.02$).

Comparison of High Performing and Low Performing Community College Students

Accuracy. A 2 (High Performing vs. Low Performing) X 8 (Problem type) ANOVA was used to examine differences in accuracy between the two subsets of Community College students by problem type. The High Performing Participants had higher overall accuracy than the Low Performing Participants (93% vs. 52%: $F(1, 17) = 132.21$; $p < 0.001$). However, there was no difference based on problem type ($F(7, 136) = 1.33$; $p = 0.24$) and no interaction between Low vs. High Performing participants and problem type ($F(7, 136) = 0.97$; $p = 0.45$).

Reaction Time. A 2 (High Performing vs. Low Performing) X 8 (Problem type) ANOVA was used to examine differences in solution time between the two subsets of Community College students by problem type. There was no difference in solution time between High and Low Performing participants (9.29 s vs. 11.04 s: $F(1, 17) = 3.22$; $p = 0.07$) and no interaction between High vs. Low Performing participants and question type ($F(7, 136) = 1.33$; $p = 0.24$). However, there was an overall difference between solution times on the different problem types ($F(7, 136) = 2.38$; $p = 0.03$).

High Performing and Low Performing participants mean solution time on the Multiply for a Common Numerator problem type were slower than the Equal Denominators problem type (12.89 s vs. 7.39 s: $F(1, 17) = 2.83$; $p = 0.005$), the Equal Numerators problem type (12.89 s vs. 7.82 s: $F(1, 17) = 2.17$; $p = 0.01$) and the Estimate Magnitudes (more than 1/3 apart) problem type (12.89 s vs. 8.32 s: $F(1, 18) = 2.35$; $p = 0.02$). Additionally, the responses to the Halves Referencing problem type were also slower than the Equal Denominators problem type (12.62 s vs. 7.39 s: $F(1, 17) = 2.69$; $p = 0.008$) and the Equal Numerators problem type (12.62 s vs. 7.82 s: $F(1, 17) = 2.47$; $p = 0.01$).

Strategies Use on Magnitude Comparison Task

The strategies were coded and analyzed in the same way as the strategies in Experiment 1. Participants in this experiment used several strategies not observed in Experiment 1; a list of these strategies can be found in Table 4. A list of strategies used in by participants in Experiment 1 and Experiment 2 can be found in Table 3.

Comparison of Strategy Use by Participants from Community College and Highly Selective University

Percent Use. A 2 (Highly Selective University vs. Community College) X 4 (Strategy type) ANOVA was used to analyze the differences in the use of each type of strategy between participants in students from the Highly Selective University and students from the Community College. There was a difference between the percent use of each strategy type ($F(3, 144) = 8.07$; $p < 0.001$) and there was an interaction between percent use for Highly Selective University students vs. Community College students and strategy type ($F(3, 144) = 14.12$; $p < 0.001$).

Overall, the strategies in the Intermediate Steps group were used more than strategies in the Logical Necessity group (37% vs. 17%: $F(1, 36) = 4.31$; $p < 0.001$) and the Usually Correct group (37% vs. 18%: $F(1, 36) = 4.08$; $p < 0.001$).

Among the students from the Community College students, students used the Questionable strategies more than the Logical Necessity strategies (41% vs. 10%: $F(1, 17) = 4.94$; $p < 0.001$). The Community College students used the Questionable strategies marginally more than the Usually Correct strategies (41% vs. 24%: $F(1, 17) = 2.68$; $p = 0.049$).

Students from the Highly Selective University used the Intermediate Steps strategy more than Community College students (49% vs. 25%: $F(1, 36) = 3.75$; $p < 0.001$). Students from the Highly Selective University also used the Logical Necessity strategies more than Community College students (25% vs. 10%: $F(1, 36) = 4.40$; $p < 0.001$). However, Community College students used the Questionable strategies more than Highly Selective University students (41% vs. 13%: $F(1, 37) = 4.40$; $p < 0.001$). The differences in the percent use of the Usually Correct strategies were non different (Highly Selective University: 13% vs. Community College: 24%: $F(1, 36)= 1.78$; $p = 0.078$).

Percent Accuracy. A 2 (Highly Selective University vs. Community College) X 4 (Strategy type) ANOVA was used to analyze the differences in the percent accuracy by strategy type between Highly Selective University students and Community College students. Consistent with the previous finding, the results indicate that Highly Selective University students were more accurate overall compared to Community College students (97% vs. 74%= $F(1, 36) = 24.47$; $p < 0.001$). Additionally, there was an overall difference between strategy types ($F(3, 129) = 10.58$; $p < 0.001$). However, there was no interaction between Highly Selective

University students vs. Community College students and strategy type in percent accuracy (F(3, 129) = 0.001; p = 1.0).

Overall, the Questionable strategies were less accurate than the other three sets of strategies: Questionable (76%) vs. Intermediate steps (95%): $F(1, 36) = 4.39$; $p < 0.001$; Questionable (76%) vs. Logical Necessity (98%): $F(1, 36) = 5.14$; $p < 0.001$; Questionable (76%) vs. Usually Correct (88%): $F(1, 36) = 2.93$; $p = 0.02$.

Among Community College students, percent accuracy when using the Questionable strategies was not lower than the Usually Correct strategies (68% vs. 78%: $F(1, 17) = 1.07$; $p = 0.37$). However, the percent accuracy when using the Questionable strategies was lower than the Intermediate Steps strategies (68% vs. 88%: $F(1, 18) = 3.58$; $p = 0.02$) and the Logical Necessity strategies (68% vs. 95%: $F(1, 17) = 6.19$; $p < 0.001$).

Additionally, comparing accuracy between Highly Selective University students and Community College students for the Questionable strategies shows that percent accuracy for Questionable strategies was higher for Highly Selective University students than for Community College students (80% vs. 68%: $F(1, 36) = 2.96$; $p = 0.004$). Percent accuracy for the Usually Correct trials was also higher for Highly Selective University students than Community College students (99% vs. 78%: $F(1, 36) = 3.52$; $p < 0.001$). Differences in accuracy in the Logical Necessity and Intermediate Steps strategies between Highly Selective University students and Community College students were statistically non different (100% vs. 95%: $F(1, 31) = 0.58$; $p = 0.45$; 98% vs. 88%: $F(1, 32) = 2.73$; $p = 0.10$).

Solution Time. A 2 (Highly Selective University vs. Community College) X 4 (Strategy type) ANOVA was used to compare solution times between the Highly Selective University students and the Community College students by strategy type. Consistent with the earlier report, solution times for Highly Selective University students were faster than solution times for Community College students (5.92 s vs. 10.12 s: $F(1, 17) = 33.29$; $p < 0.001$). Additionally, there were differences between solution times on the different strategy types ($F(3, 129) = 7.42$; $p < 0.001$). Comparisons reveal that solution times on the Questionable Strategies were slower than the Logical Necessity strategies (10.15 s vs. 5.04 s: $F(1, 31) = 7.25$; $p < 0.001$). Differences between the solution times for the Immediate Steps strategies (mean = 7.20 s, *SD* = 5.09) and the other strategies and the Usually Correct strategies (mean = 7.35 s, *SD* = 6.43) and the other strategies were statistically non different.

Among Community College students, the Questionable strategies were slower than the Logical Necessity strategies (11.93 s vs. 6.25 s: $F(1, 17) = 3.40$; $p = 0.009$). All other differences in solution times were non different. The mean solution times were as follows: Logical Necessity (6.25 s, *SD* = 2.69), Intermediate Steps (9.91 s, *SD* = 6.40), Usually Correct (10.63 s, *SD* = 7.57), and Questionable (11.93 s, *SD* = 5.76).

Comparisons among the solution times for Highly Selective University students and Community College students by strategy type show that for all of the strategies except for Logical Necessity, solution times for the Highly Selective University students were shorter than solution times for Community College students: Intermediate Steps (5.27 vs. 5.20: $F(1, 32) = 2.93$; $p = 0.004$); Usually Correct (4.35 s vs. 10.63 s: $F(1, 36) = 4.25$; $p < 0.001$); and Questionable strategies (8.62 s vs. 11.93 s: $F(1, 36) = 2.42$; $p = 0.02$). Solution times for the Logical Necessity strategy group did not differ (4.32 s vs. 6.25 s: $F(1, 36) = 1.24$; $p = 0.22$).

Strategies of High Performing and Low Performing Community College Students

For the High Performing participants, the Intermediate Steps strategies were used the most (43%). The logical necessity strategies were used the least at 15% and the other two categories were both close to 20% (Figure 4). The Questionable strategies had the lowest accuracy (87% *SD* = 15). The other three sets of strategies were all near 95% accuracy (Figure 5). The Questionable strategies (mean ST = 11.23 s) and the Intermediate Steps strategies (mean ST = 10.51 s) had the slowest solution times compared to the other 2 categories. The other two sets of strategies had mean solution times at 7.45 s for Usually Correct strategies and 6.18 s for logical necessity strategies (Figure 6).

For the Low Performing participants, the Questionable strategies were used the most (64%). The logical necessity strategies and the Intermediate Steps strategies were both used 4% of the time while the Usually Correct strategies were used 28% of the time (Figure 4). The Questionable strategies had the lowest accuracy (mean = 46% *SD* = 21) compared to the other 3 sets of strategies. The logical necessity strategies were the most accurate (mean = 87% *SD* = 27). The Intermediate Steps strategies had a mean accuracy of 72% (*SD* = 30) and the Usually Correct strategies had a mean accuracy of 58% (*SD* = 20). The Questionable strategies had the slowest solution time (mean = 12.20 s) compared to the other three sets of strategies. The logical necessity strategies had a mean solution time of 7.67 s and Intermediate Steps strategies had a mean solution time of 7.25 s. The Usually Correct strategies had a mean solution time of 9.09 s (Figure 6).

A 2 (High Performing vs. Low performing) X 4 (Strategy type) ANOVA was used to examine differences in the percent use of each type of strategy by participant performance level. The results demonstrate that there was an overall difference between use of each strategy type across participants ($F(3, 68) = 7.54$; $p < 0.001$). The Questionable strategies were used more than the Logical Necessity strategies ($F(1, 17) = 7.52$; $p < 0.001$). There was also a interaction between percent use of type of strategy and participant performance ($F(3, 68) = 13.65$; p <

0.001). High Performing participants used Intermediate Steps strategies more than the Low Performing participants (44% vs. 4%: $F(1, 17) = 4.17$; $p < 0.001$). Conversely, Low Performing participants used the Questionable strategies more than the High Performing participants (64% vs. 21%: $F(1, 17) = 4.63$; $p < 0.001$). Differences in use of the logical necessity strategies (High: 15% vs. Low 3.5%: $F(1, 17) = 1.24$; $p = 0.22$) and the Usually Correct strategies (High: 21% vs. Low: 28%: $F(1, 17) = 0.78$; $p = 0.44$) were non different.

Percent Accuracy. In order to compare differences in percent accuracy of each strategy type among the High and Low Performing participants a 2 (High Performing vs. Low Performing) X 8 (Strategy type) ANOVA was used. There was an overall difference based on strategy type consistent with earlier reports ($F(3, 56) = 15.43$; $p < 0.001$) and there was also a difference in accuracy based on High and Low Performing participants (93% vs. 52%: $F(1, 17) = 132.21$; $p < 0.001$). The High Performing participants had higher accuracy for each of the strategy types compared to the Low Performing participants (Logical Necessity: 100% vs. 87%: $F(1, 17) = 4.56$; $p = 0.002$; Intermediate Steps: 94% vs. 6%: $F(1, 17) = 5.67$; $p = 0.003$; Usually Correct: 93% vs. 58%: $F(1, 17) = 9.45$; $p < 0.001$; Questionable: 87% vs. 46%: $F(1, 17) = 7.98$; $p < 0.001$)

Solution Time. In order to compare differences in solution time of each strategy type among the High and Low Performing participants a 2 (High Performing vs. Low Performing) X 8 (Strategy type) ANOVA was used. The results did not indicate any differences or interaction: (High: 9.38 s vs. Low 10.98 s: $F(1, 17) = .72$; $p = 0.40$); (Strategy type: $F(3, 56) = 2.25$; $p = 0.092$); (interaction $F(3, 56) = 0.02$; $p = 1.0$). The lack of difference is probably due to the low number of observations of each and the wide variance in times within the observed trials. For example, the Low Performing participants used the Intermediate Steps trials about 4% of the time and had a mean solution time of 7.25 s and a standard deviation of 7.08 s.

Comparison of Strategy Use from Students at the Community College and the Highly Selective University

A 2 (Highly Selective University vs. High Performing Community College) X 4 (Strategy type) ANOVA was used to analyze the differences in percent use, percent accuracy, and solution time. The High Performing Community College participants and the students from the Highly Selective University generated a similar pattern of results in the percent use of each type of category. There was no interaction among percent use of different strategies between Highly Selective University students and High Performing Community College students ($F(3, 108)$: 2.31; $p = 0.09$).

Highly Selective University students were more accurate overall than High Performing Community College students (97 vs. 93: $F(1, 26) = 46.56$; $p < 0.001$). Additionally, there was an interaction between strategy type and Highly Selective University vs. High Performing Community College students ($F(3, 104) = 45.78$; $p < 0.001$). For the Questionable strategies, Highly Selective University students were more accurate than High Performing Community College students ($F(1, 26) = 13.56$; $p < 0.001$). However, there were no differences in accuracy of any of the other strategy types.

Highly Selective University students were faster than High Performing Community College students overall (5.92 vs. 9.38: $F(1, 26) = 25.52$; $p < 0.001$). However, there was no interaction between participant type and strategy ($F(3, 104) = 1.35$; $p = 0.26$). Solution times for the Intermediate Steps strategies were faster for the Highly Selective University students compared to the High Performing Community College students (5.27 vs. 10.51: $F(1, 26) = 10.49$; $p < 0.001$).

A 2 (Highly Selective University vs. Low Performing Community College) X 4 (Strategy type) ANOVA was used to analyze the differences in percent use, percent accuracy, and solution time of the different strategies between the two groups. There was an overall difference in percent use of the strategies ($F(3, 104) = 8.96$; $p < 0.001$) and an interaction ($F(3, 104) = 48.62$; $p < 0.001$). The Low Performing Community College students used Questionable strategies more than Highly Selective University students (Highly Selective University: 13% vs. Low Community College: 64%: $F(1, 25) = 45.82$, $p < 0.001$), and used Usually Correct strategies more than Highly Selective University students (Highly Selective University: 13% vs. Low Community College: 28%: $F(1, 25) = 5.45$, $p = 0.028$). Highly Selective University students used both the Logical Necessity strategies and the Intermediate Steps strategies more than the Low Performing Community College students (Highly Selective University: 25% vs. Low Community College: 4%: $F(1, 25) = 69.16$, $p < 0.001$; Highly Selective University: 49% vs. Low Community College: 4%: $F(1, 25) = 50.97$, $p < 0.001$).

The Low Performing Community College students were less accurate overall than the Highly Selective University students (52 vs. 97: $F(1, 25) = 45.62$; $p < 0.001$). The Low Performing Community College students were less accurate than the Highly Selective University students for each of the different sets of strategies (Logical Necessity: 87% vs. 100%: $F(1, 21) = 5.41$, $p = 0.03$); Intermediate Steps: 72% vs. 98%: $F(1, 22) = 20.90$, $p < 0.001$); Usually Correct: 58% vs. 99%: $F(1, 23) = 41.52$, $p < 0.001$); Questionable: 45% vs. 80%: $F(1, 26) = 15.04$, $p < 0.001$).

The Low Performing Community College students were also slower overall than the Highly Selective University students (10.98 vs. 5.92: $F(1, 25) = 34.23$; $p < 0.001$). The Low Performing Community College students were slower than the Highly Selective University students for each set of strategies (Logical Necessity: 7.67 s vs. 4.32 s: $F(1, 21) = 15.56$, $p < 0.001$); Usually Correct: 9.09 s vs. 4.35 s: $F(1, 23) = 8.56$, $p = 0.008$); Questionable: 12.25 s vs. 8.62 s: $F(1, 26) = 4.10$, $p = 0.49$) except for the Intermediate Steps strategies which had a non-difference such that the Low Performing Community College students were slower than the Highly Selective University students (7.25 s vs. 5.27 s: $F(1, 22) = 3.38$, $p = 0.79$)).

Discussion

The Experiment 2 results demonstrated that there is a wide range of strategies used in fraction magnitude comparisons and that these strategies are related to internal magnitude representation of fractions and overall mathematical understanding. The results show that performance on the magnitude comparison task varied greatly among the Community College students. The High Performing Community College students had very accurate magnitude representations as demonstrated by their low PAE results on the number line estimation task. However, the Low Performing Community College students had a much higher PAE on the number line estimation task.

The High Performing Community College students mirrored the students at the Highly Selective University in the types of strategies that were used. The two groups used Logical Necessity strategies and Intermediate Steps strategies more often than the Usually Correct and the Questionable strategies. The central difference was in the solution times of the two groups of participants for the Intermediate Steps strategies. High Performing Community College students were the slowest for this set of strategies (slower than the Low Performing Community College students and the Highly Selective University students). This may be because the High Performing Community College students were actually executing each of the steps but it takes them longer to do so than the Highly Selective University students. However the Low Performing Community College students were probably not executing each of the steps correctly or were skipping over steps and thus got the wrong answer.

This speculation is in line with Stigler, Givven, and Thompson (2010)'s results that show the common errors that community college students make when performing computations with fractions. Many times they make simple computational errors such as simplifying the fractions incorrectly before multiplying the fractions. This might result in a slowed non-automatic approach to solving the problem. In the current study, these were High Performing Community College students who were probably able to solve the problem correctly but might have to search through a variety of possible strategies and algorithms and make sure that they are executing each step methodically.

The Low Performing Community College students used the Usually Correct strategies and Questionable strategies the most. This is probably why these participants were the least accurate. Even when the participants used Usually Correct strategies, the High Performing Community College students and the students from the Highly Selective University still had accuracies near 100% but the Low Performing Community College students' accuracies were closer to 60%. Additionally, the Low Performing Community College students were slowest except for when they used the Intermediate Steps strategies. This demonstrates that even though they spent more time considering the problem, they did not reach more accurate results. Thus, it seems that they had a difficult time either deciding which strategy would be appropriate for the specific comparison, or that they knew the necessary steps but had difficulty executing them.

One question that this led us to ask is why the Low Performing Community College students would use such unsuccessful strategies so often. Did these participants realize that the strategies were inaccurate, or could they not think of better alternatives? In order to investigate this, Experiment 3 looked at whether community college students would choose more successful strategies if such strategies were explicitly presented as options.

Experiment 3

Methods

Participants

Participants were 20 community college students (12 female, median age 19) recruited from introductory math courses. The mean SAT math scores for the 5 participants who reported their scores was 513. The scores ranged from 300 to 650.

Design

This study was a within participants design with the same 8 problem types as in Experiments 1 and 2. However, in this study, participants were given the option to change their strategy choice to either a bad alternative, a good alternative, or stay with the same strategy. The strategies that were given as bad alternatives were derived from poor strategies that participants used in Experiment 2 in the Questionable strategies category. The good alternatives were derived from the strategies used in Experiments 1 and 2. Good alternatives were taken mainly from the Logical Necessity and Intermediate Steps strategy groups. However, a small portion of the strategies was also taken from the Usually Correct strategies. When these were used as good alternatives, they were used on problems where they would result in the correct answer. The bad alternatives were taken from the Questionable strategy group and the Usually Correct strategies. In each case, the good alternative would always result in the correct answer while the bad alternative would always result in the incorrect answer.

The study used 32 magnitude comparison trials, with 4 comparisons per problem type.

Procedure

The first block of problems used the same procedure as was used in Experiments 1 and 2. The participants were given a magnitude comparison problem, asked to estimate which fraction was larger, and then asked to describe their strategy. The strategy explanations were coded and stored online during the experiment.

The second block of problems consisted of the same 32 magnitude comparisons from block 1. However, during this block, participants were shown the problem and three possible strategies; they were instructed to choose the strategy they thought would best solve the problem. One of the strategies was the same as the one that they had used to solve the problem during block 1. A second strategy was a bad alternative and a third was a good alternative. Depending on whether their original strategy was good or not, there could be two strategies listed that would give the correct answer or two strategies that could yield the incorrect answer. Participants were not told which of the strategies was the same as their original strategy. They were told that one of them might be the same as their original strategy but that they should not use that as a basis to choose which strategy to pick. Rather, they should pick whichever strategy they think would be the best of the 3 to use to solve the problem. After selecting the strategy, participants were asked to explain their answers. See Table 5 for examples of the alternatives.

Results

Magnitude Comparison

On the original magnitude comparison task, participants had an average accuracy of 85% (*SD* = 15). Logical Necessity strategies were used on 14% of trials, Intermediate Steps strategies on 41%, Usually Correct strategies on 25%, and Questionable strategies on 16% (Figure 4).

Accuracy. A one way ANOVA was used to examine the differences in percent accuracy for the Community College students. The Questionable strategies were less accurate that the other three types of strategies (mean = 64%, vs. 99%, 93%, and 94%, respectively, for Logical Necessity strategies, Intermediate Steps strategies, and Usually Correct strategies $F(3, 61)= 6.54$, $p < .001$)) (Figure 5).

Strategy Switching. When participants originally used Logical Necessity strategies, they stayed with their strategies (80% of trials) more than when they originally used any of the other three sets of strategies ($F(3, 60) = 4.98$; $p = 0.004$). When participants originally used Intermediate Steps strategies, they stayed with their strategies more than the Questionable strategies (68% vs. 49%: $F(2, 29) = 3.97$; $p = 0.006$). When the Usually Correct strategies were used first, participants stayed with their original strategy less than when the participants used Questionable strategies originally (17% vs. 49%: $F(2, 33) = 3.58$; $p = 0.006$). Table 6 shows the percent of over all trials that participants either did not change, changed to good strategies, or changed to bad strategies separated by original strategy type. Table 7 shows a matrix of original strategies by final strategies categorized by the four strategy types.

Strategy switching was also associated with overall accuracy on the task. Correlations were used to examine the relationship between the percent of all trials where participants switched to a good (or bad) alternative and overall accuracy on the task. The less accurate participants' magnitude comparisons were initially, the more often they switched to bad alternative strategies (*r* = -.64; *t* = -3.5; p = 0.002). However, the percent of trials on which participants switched to good alternatives was uncorrelated with overall accuracy (*r* = 30; *t* = 1.18; p = 0.25; *r* = 0.09; *t* = 0.41; p = 0.7) (Figure 7). To interpret this result, it is important to consider that many participants started with good strategies and so did not switch.

Additionally, the percent of trials on which participants changed from bad strategies to bad alternatives was also correlated with overall accuracy (*r* = -.67; *t* = -3.7; p = 0.002). Of the times that participants started out with good strategies, the percent of times that they do not change their strategies is , positively correlated with overall accuracy (*r* = .49; *t* = 2.34; p = .032). And, when participants started out with a good strategy, the percent of trials on which they changed to bad alternatives was negatively correlated with overall accuracy (*r* = -0.59; *t* = -3.01; *p* = 0.008).

Discussion

The central conclusion from this experiment involves the results of the strategy switching portion of the study. When participants switched to new strategies, they often chose strategies that would be good alternatives. The participants who chose poor alternatives tended to be participants who were doing poorly on the task overall.

Additionally, participants were most likely to switch strategies when they used Usually Correct strategies. This result is interesting because it demonstrates that participants were using less optimal strategies that might work some of the time and instead chose to switch to strategies that would be more reliably accurate. Conversely, participants were less likely to change their strategies when they used Questionable strategies, which might suggest that participants who are using these strategies are less cognizant of their poor strategy choice.

General Discussion

The studies reported here replicate previous findings that when problems can be solved by componential strategies, adults use componential strategies (Bonato et al, 2007; Kallai & Tzelgov, 2009). However, when they cannot use these strategies, they use strategies that rely on evaluating how the integrated magnitudes of the two fractions compare (Schneider & Siegler, 2010l; Meert, Gregoire, Noel, 2009; Ischebeck et al.). A large percentage of the time, participants cited a "General Magnitude Reference" strategy or "Halves Referencing" strategy. These were most often used on problems where one of the more obvious componential strategies like Equal Denominators could not be used.

These experiments contribute to the existing literature because they are the first to show both componential and integrated magnitudes within a single set of problems. Further, they enumerate the various types of strategies that people use to deal with these types of fraction magnitude comparisons. They also demonstrate that there are large individual differences in the choices of strategies. The differences in strategy choice vary with the differences in overall performance on the magnitude comparison task and number line estimation ability suggesting that those with high mathematical knowledge most often use the better strategies and those who have a less robust understanding of fraction magnitude and mathematical understanding use the poorer strategies.

These three experiments shed light on what types of strategies adults use in fraction magnitude comparisons, how these strategies vary with overall mathematical knowledge, and how adults adaptively change to strategies that are more reliable. Among adults who have a well-developed mathematical background, the strategies most often used are those that would be correct by Logical Necessity, either with the problem as stated or with one or two Intermediate Steps. Adults with good mathematical backgrounds also sometimes use strategies that are less reliable, but they obtain accurate results with these strategies, which suggests that they are selectively using these strategies on problems where the strategies are beneficial.

This ability to selectively use strategies in the appropriate circumstances to yield correct results is related to overall mathematical background knowledge. In Experiment 2, Low Performing Community College students who performed poorly on the magnitude comparison task were more likely to use strategies that would yield incorrect results and did not seem to be fully aware of more reliable and beneficial strategies. Also related to this, these Low Performing Community College students also had very low accuracy or high PAE on the number line estimation task.

It seems that this faulty underlying magnitude representation may be related to the use of poor strategies on the task. If these participants had a better magnitude representation of the fractions, they might be able to use this as a check to identify faulty strategies during the magnitude comparison task. For example, if the strategy that they were using yielded an answer that seemed contrary to their internal representation of the value, then they might revise the strategy or perhaps not use it again. Enhancing magnitude representations has been shown to increase arithmetic performance in children as they notice bugs in their arithmetic strategies and thus revise their strategy use (Siegler & Ramani, 2008).

The results of Experiment 3 further demonstrate the intricacies of the strategy selections of community college students. There was a negative correlation between the percent of time participants switched to a bad alternative and overall accuracy on the original magnitude comparison task. This suggests that the participants who did better on the original task either stayed with their original strategy or switched to a good alternative. Additionally, there was a negative correlation between switching from one bad strategy to another bad strategy and overall accuracy on the original task. This further suggests that participants who used Questionable strategies originally were unsuccessful at selecting alternative strategies that would be advantageous.

Additionally, there was a positive correlation between overall accuracy on the original task and the percent of trials on which good strategies were used and were not later changed. This suggests that participants who performed better on the task originally used good strategies and were confident in their choices. They did not end up switching to a bad or a good alternative. Along with this, strategies that must be correct either with sufficient information or when steps are executed correctly have the lowest switch rates.

Usually Correct strategies had the highest rate of switching to better alternatives. This suggests that participants who used these strategies were using them in instances where there was an alternative that was clearly better or clearly more successful. Further, participants who used strategies that are not Usually Correct stayed with their original strategy on 59% of trials, which suggests that they many such participants either did not understand the advantages of alternative approaches or understood those advantages only sporadically.

Another interesting finding, from Experiment 2, is that the High Performing Community College students were slower with the Intermediate Steps strategies than the Low Performing

Community College students in Experiment 2 and the adults from the Highly Selective University in Experiment 1. This highlights how information processing and working memory capacity also play a role in the correct execution of a series of steps within a particular strategy. Information processing and working memory abilities were probably highest among the students at the Highly Selective University, based on the results of the collected SAT scores. However, performance on the magnitude comparison task for the High Performing Community College students was not too much lower than those from the Highly Selective University. The difference in solution time was the main marker between the two sets of participants. In addition, the Low Performing Community College students who used strategies that must be correct by Logical Necessity when the steps are executed correctly did so less accurately and quickly than the High Performing Community College students. This finding suggests that the High Performing Community College students were taking the time to go through the steps but were just not as quick as Highly Selective University students.

In all, this study sheds some light onto the debate between how adults process and represent fraction magnitudes during magnitude comparison tasks. It provides evidence that adults use different strategies depending on the context of the magnitude comparison. In some cases, adults compare magnitudes, while in others, they use componential strategies that do not necessarily involve accessing a magnitude representation for the fraction. Additionally, the ability to appropriately use these strategies seems to depend on overall mathematical knowledge and understanding of fractions. When given the opportunity to use more appropriate strategies, community college students who used good strategies might switch to better ones but those who used the poorest strategies rarely switched to better ones. This suggests that an overall global understanding of fractions and number is important for successfully selecting and applying strategies to compare fraction magnitudes.

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Tables and Figures

Tables

Table 1: Examples of fraction stimuli used in the magnitude comparison task.

Problem Type	Larger Fraction	Smaller Fraction
Equal Denominator	3/7	2/7
	13/17	9/17
Equal Numerator	3/4	3/5
	2/13	2/17
Estimate Magnitudes (less than 1/3 apart)	3/4	5/9
	5/12	6/19
Estimate Magnitudes (more than 1/3 apart)	4/9	1/8
	9/19	2/17
Halves Referencing	2/3	3/7
	11/16	6/13
Larger Numerator and Smaller Denominator	3/7	2/9
	3/11	2/15
Multiply for Common Denominator	2/3	5/9
	3/8	5/16
Multiply for Common Numerator	4/5	2/3
	3/7	5/14

Table 2: Highly Selective University Students (Experiment 1) and High and Low Performing Community College Students (Experiment 2) magnitude comparison task mean accuracy and mean solution time by problem type

Problem Type	Mean Accuracy (%)			Mean Solution Time (s)		
	Experiment 1: Highly Selective University	Experiment 2: High Performing Community College	Experiment 2: Low Performing Community College	Experiment 1: Highly Selective University	Experiment 2: High Performing Community College	Experiment 2: Low Performing Community College
Equal Denominator	100	97	61	2.93	4.40	10.72
Equal Numerator	98	97	65	4.49	6.03	10.50
Estimate Magnitudes (less than 1/3 apart)	93	95	46	6.99	11.05	11.12
Estimate Magnitudes (more than 1/3 apart)	98	98	41	4.19	6.65	10.83
Halves Referencing	93	87	57	7.74	13.83	11.28
Larger Numerator and Smaller Denominator	100	98	54	3.55	10.50	11.28
Multiply for Common Denominator	97	87	43	5.85	8.69	11.37
Multiply for Common Numerator	89	88	48	8.41	13.78	11.90

Table 3: Strategies used during fraction magnitude comparison by participants in Experiment 1

General Strategy Group	Strategies Included	Strategy Description
"Logical Necessity Strategies": Strategies that must be correct by logical necessity - when given sufficient information	Larger numerator and smaller denominator	The larger fraction has a larger numerator and a smaller denominator than the smaller fraction.
	Equal numerators	If both fractions have equal numerators, the fraction with the smaller denominator is larger.
	Equal denominators	If both fractions have equal denominators, the fraction with the larger numerator is larger.
"Intermediate Steps Strategies": Strategies that must be correct- when intermediate steps executed correctly	Multiply for a common denominator	Multiply one fraction by 1 (in the form of fraction a/a) in order to get common denominators.
	Multiply for a common numerator	Multiply one faction by 1 (in the form of fraction a/a) in order to get common numerators.
	Halves referencing	The larger fraction is greater than ½ and the smaller fraction is smaller than ½.
	Convert to decimal/percent	Covert the fraction into its decimal or percent form and compare the values in those forms.
	General magnitude reference	Compare the magnitudes in reference to a nearby known magnitude like 0, 1/3, 1/2, 2/3, 3/4, or 1.
"Usually Correct Strategies": Strategies that should yield better than chance results but that aren't necessarily correct	Visualization	Using a pie, pizza, or other visual representation of a fraction in order to compare magnitudes
	Difference between numerator and denominator is smaller	The difference between the numerator and denominator of the larger fraction is smaller than the difference between the numerator and denominator of the smaller fraction.
"Questionable Strategies": Strategies not guaranteed to yield above chance performance	Guess and other	Other includes strategies that were used less than 3% of the time across all trials. Strategies were marked as "guess" when the participant explicitly stated that s/he guessed on the given trial.

Table 4: Additional strategies used in fraction magnitude comparison by Community College Students (participants in Experiment 2) but not used by Highly Selective University Students (participants in Experiment 1)

General Strategy Group	Strategies Included	Strategy Description
"Usually Correct Strategies": Strategies that should yield better than chance results but that aren't necessarily correct	Bigger numerator	The fraction with the bigger numerator is larger.
	Smaller denominator	The fraction with the smaller denominator is larger.
"Questionable Strategies": Strategies not guaranteed to yield above chance performance	Bigger numerator and denominator	The fraction with the bigger numerator and denominator is larger
	Smaller numerator and denominator	The fraction with the smaller numerator and denominator is larger
	Larger remainder when denominator is divided by numerator	The fraction with the larger remainder when the denominator is divided by the numerator is larger.
	Denominator divided by numerator is larger	The fraction with the larger quotient when the denominator is divided by the numerator is larger

Table 5: Examples of Alternative strategies used in Experiment 3

Good Alternatives	Bad Alternatives
Convert to decimal/percent	The larger fraction has smaller numerator and denominator than the other fraction
Equal denominators	The larger fraction has a bigger denominator than the other fraction
Equal numerators	The larger fraction has a smaller denominator than the other fraction
General Magnitude reference (around values like 0, ¼, 1/3, ½, 2/3, ¾, 1)	The larger fraction has a bigger numerator and bigger denominator than the other fraction
Halves referencing	The denominator divided by the numerator is larger than the other fraction
Larger numerator and smaller denominator	The larger fraction has a larger remainder when the denominator is divided by the numerator
Multiply for a common denominator	
Multiply for a common numerator	
Cross multiply for a common denominator	
Difference between the numerator and denominator is smaller than the difference in the other fraction	
Denominators close, pick larger numerator	

*Note: When offered as alternatives, the good alternative always yielded the correct answer and the bad alternative always yielded the incorrect answer.

Table 6: Percent of times participants did not change their original strategy or changed to a good alternative or to a bad alternative during Experiment 3.

Strategy	Percent of times did not change	Percent of times changed to good	Percent of times changed to bad
Logical Necessity	80	17	3
Intermediate Steps	68	27	5
Usually Correct	17	68	15
Questionable	49	34	17

Table 7: Distribution original and final strategy choice by strategy type (shown as percents) for Experiment 3.

Original Strategy	Final Strategy			
	Logical Necessity	Intermediate Steps	Usually Correct	Questionable
Logical Necessity	80	12	5	3
Intermediate Steps	7	88	3	2
Usually Correct	21	28	45	6
Questionable	9	26	7	59

Figures

Figure 1: Number line estimation task percent absolute error for Highly Selective University Students (Experiment 1), High Performing Community College Students, Low Performing Community College Students by participant

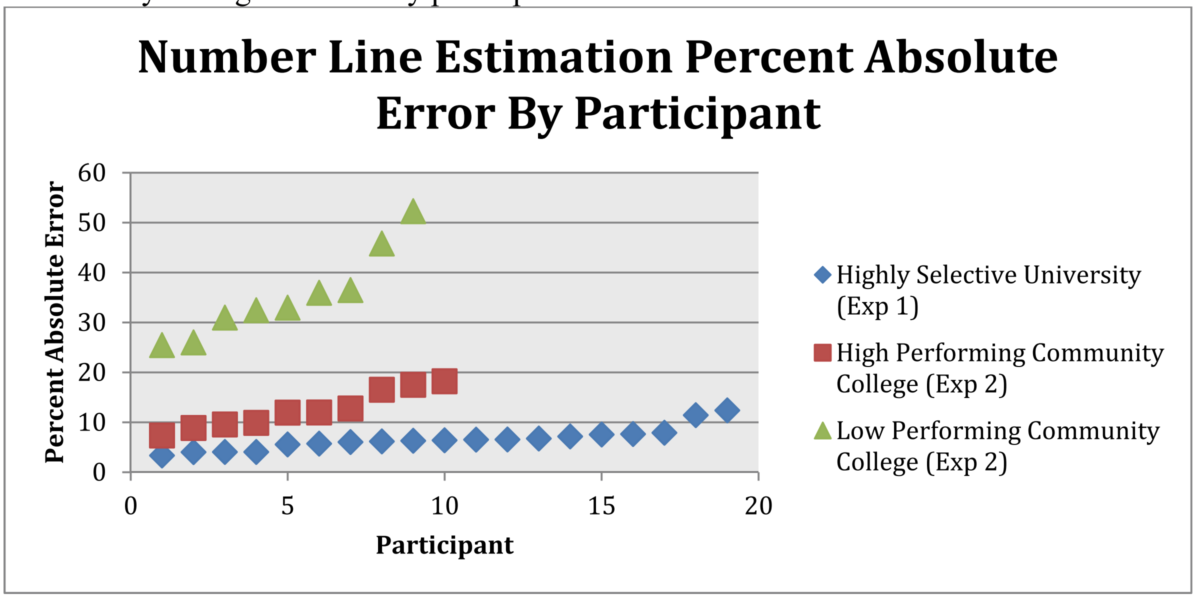

Figure 2: Number line estimation task mean estimate vs. actual magnitude for Highly Selective University Students (Exp 1), High Performing Community College Students (Exp 2), Low Performing Community College Students (Exp 2). Linear regression equations are shown.

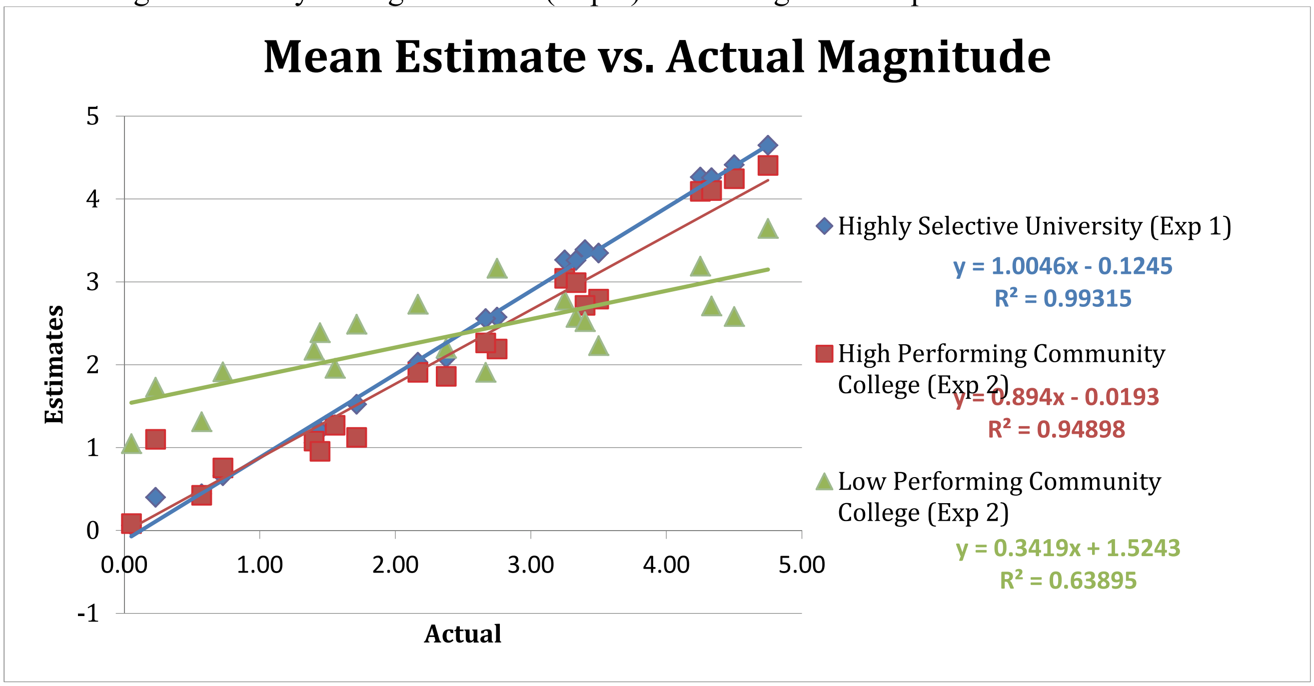

Figure 3: Percent accuracy for magnitude comparison task by participant for Experiments 1, 2, and 3.

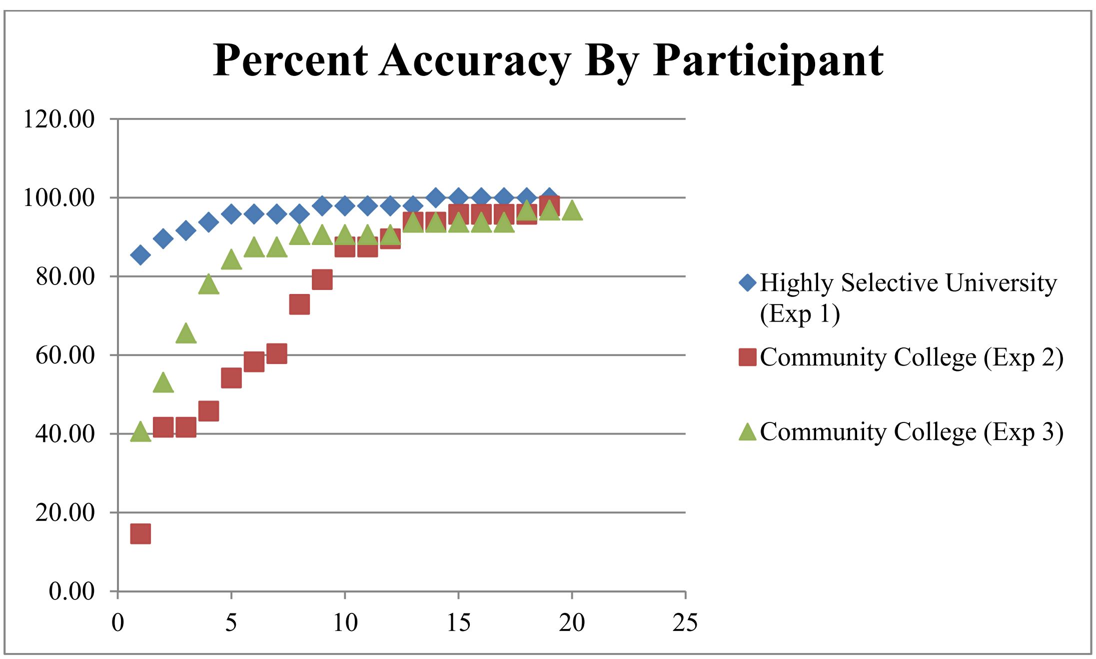

Figure 4: Percent of types of strategies that were used in Experiments 1, 2, and 3

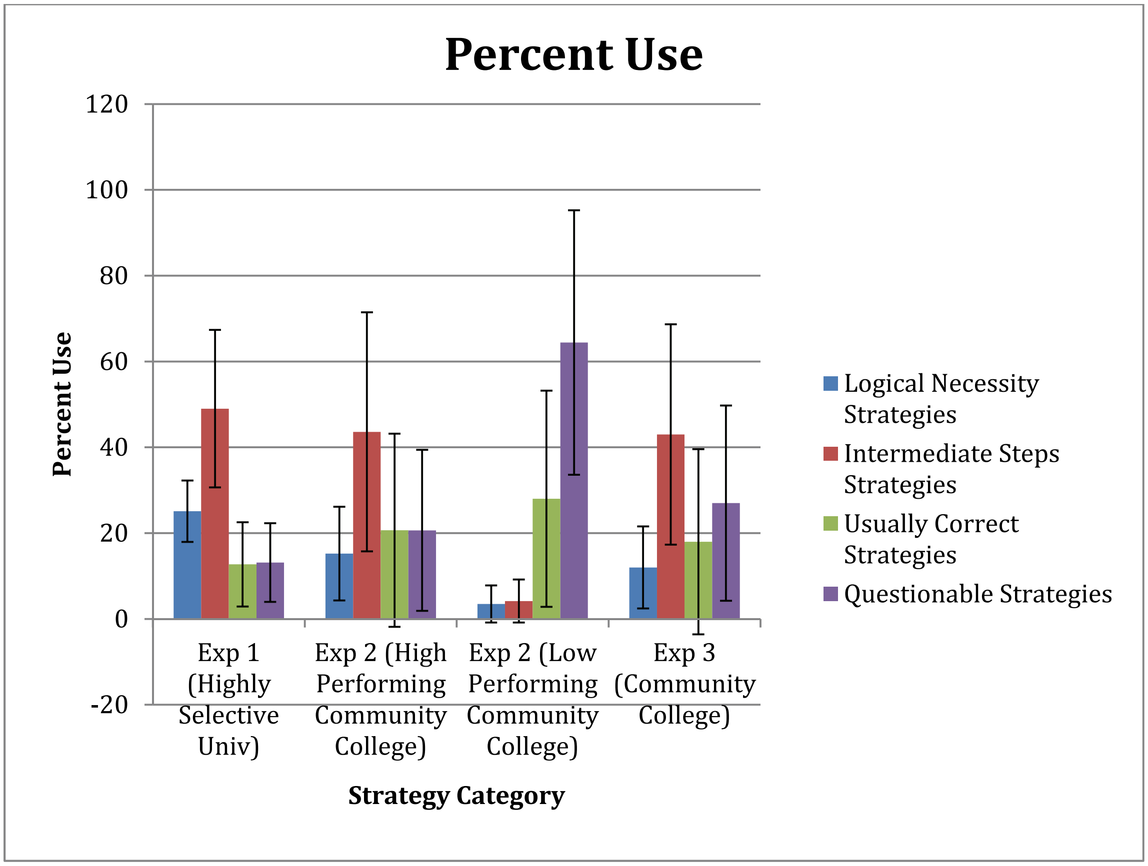

Figure 5: Percent accuracy by types of strategies used for Experiments 1, 2, and 3

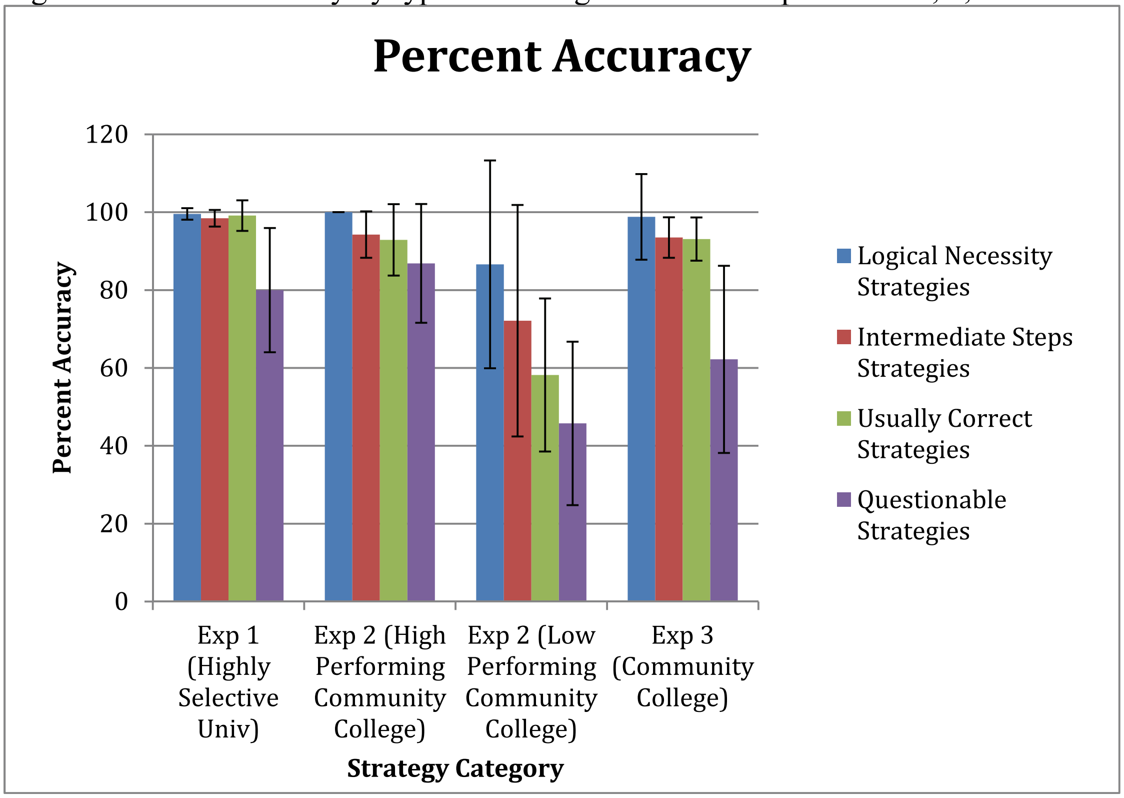

Figure 6: Solution time by types of strategies used for Experiments 1, 2, and 3

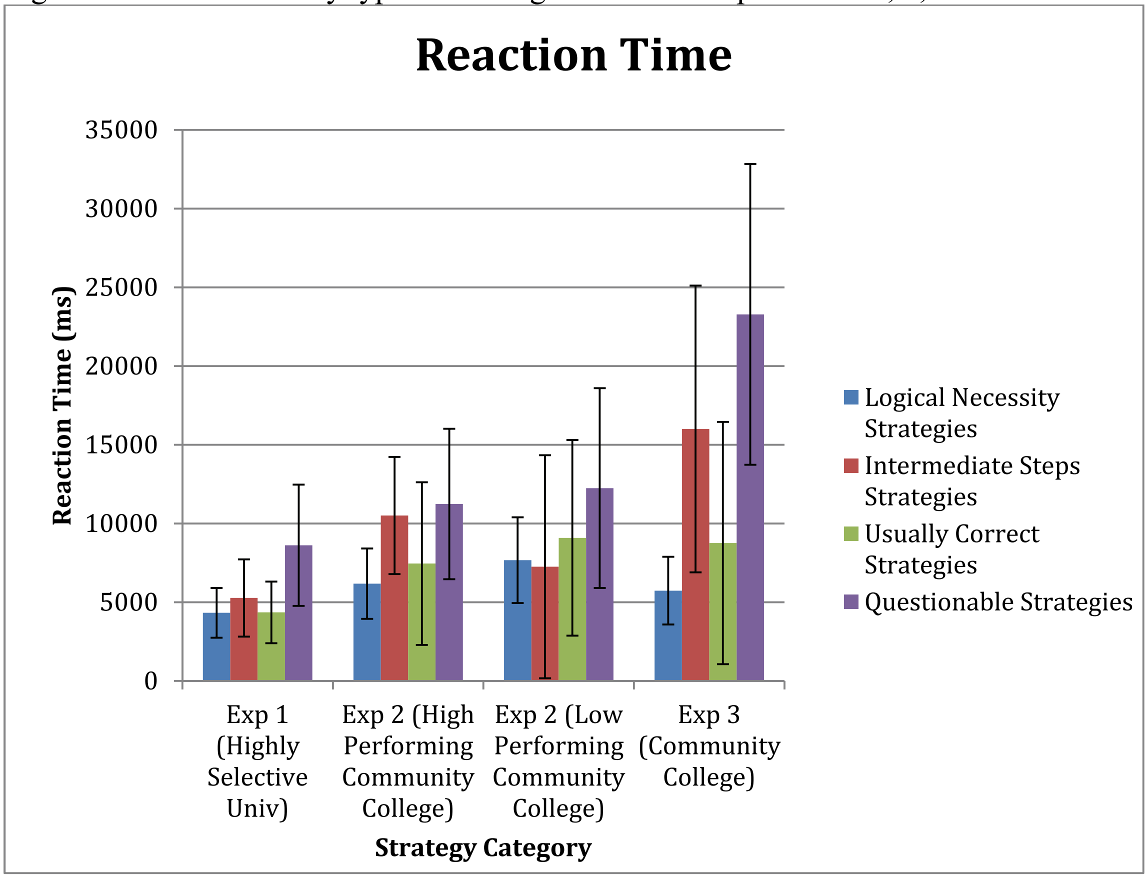

Figure 7: Percent of all trials switched strategies to good or bad alternatives vs. overall accuracy on magnitude comparison task for Experiment 3. Linear regression equations are shown with equations.